# UNIVERSITY COLLEGE LONDON

## EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATH2101**

MODULE NAME : **Analysis 3: Complex Analysis**

DATE : **08-May-07**

TIME : **10:00**

TIME ALLOWED : **2 Hours 0 Minutes**

2006/07-MATH2101A-001-EXAM-196

*All questions may be attempted but only marks obtained on the best* **four** *solutions will count.*
*The use of an electronic calculator is* **not** *permitted in this examination.*

1. (a) Let $f$ be a complex valued function on a domain $D \subset \mathbb{C}$. Define each of the following:
   (i) $f$ is holomorphic on $D$
   (ii) $f$ has a root of multiplicity $n$ at a point $z_0 \in D$.

   (b) Give an example of
   (i) a function $f$ holomorphic everywhere except at $\pm 1$
   (ii) an entire function $f$ for which $1/f$ fails to be holomorphic at precisely three points
   (iii) a function $f$ which is not holomorphic anywhere.

   You do not need to justify your answers.

   (c) Let $f$ be a holomorphic function of $z = x + iy$ where $x$ and $y$ are real, and let $u = \operatorname{Re} f$, $v = \operatorname{Im} f$. State and prove the Cauchy - Riemann equations for $f$.

   (d) Show that $f$ defined by

$$f(z) = \begin{cases} \frac{z^5}{|z|^4} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

   satisfies the Cauchy-Riemann equations at $z = 0$ but it is not differentiable there.
   Hint: calculate the derivative of $f$ along lines passing through 0.

2. (a) Define the functions $e^z$, $\cos z$ and $\sin z$ for $z \in \mathbb{C}$.

   (b) Which of the following statements are true?
   (i) $e^{z+w} = e^z + e^w$ for all $z, w \in \mathbb{C}$.
   (ii) $e^z \neq 0$ for any $z \in \mathbb{C}$.
   (iii) $|\sin z| \leq 1$ for all $z \in \mathbb{C}$.
   (iv) $\cos^2 z + \sin^2 z = 1$ for all $z \in \mathbb{C}$.

   You do not need to justify your answers.

   (c) State Euler's formulas and use them to prove that $\cos 2z = 2\cos^2 z - 1$.

   (d) Let $z = x + iy$, where $x$ and $y$ are real. Write the real and imaginary parts of $e^{2z}$, $e^{z^2}$ and $e^{e^z}$ as an expression of $x$ and $y$.

3. (a) Define what we mean by the radius of convergence of a power series $f(z) = \sum_{n=1}^{\infty} a_n z^n$.

(b) For each of the following power series, calculate the radius of convergence and hence find at which points the sum of the power series defines a holomorphic function:

(i) $\sum_{n=1}^{\infty} \frac{(-2)^n z^n}{n^3}$

(ii) $\sum_{n=1}^{\infty} \frac{z^n}{n^n}$

(iii) $\sum_{n=1}^{\infty} z^{2n}$.

(c) Show that the radius of convergence of $f(z) = \sum_{n=1}^{\infty} a_n z^n$ is

$$R = \sup\{|z| : a_n z^n \to 0\}.$$

Deduce that $f(z)$ and $f'(z)$ have the same radius of convergence.
Hint: recall the proof of $1/R = \limsup \sqrt[n]{|a_n|}$.
You may use the power series expansion $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$ without proof.

4. (a) Prove the following version of Cauchy's theorem: if $f$ is holomorphic on a domain $D$, and if $\gamma \subset D$ is a piecewise smooth simple closed curve whose interior also belongs to $D$, then

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z}\, dw$$

for all $z$ which is in the interior of $\gamma$.
You may use without proof that $\int_{\gamma} f(z)\, dz = 0$ for any holomorphic function $f$ on $D$.

(b) Evaluate

(i) $\int_{\gamma} \frac{z^5 + i}{z - i}\, dz$

(ii) $\int_{\gamma} \frac{e^{z^3}}{(z-1)^3}\, dz$

where $\gamma$ is a circle with centre 0 of radius 2.

5. (a) Define each of the following:
   (i) $f$ has a removable singularity at a point $z$
   (ii) $f$ has a pole of order $m$ at a point $z$
   (iii) $f$ has an essential singularity at a point $z$.

   (b) Determine whether the following functions have a removable singularity, pole or essential singularity at $z = 0$:
   (i) $\frac{1}{z+\frac{1}{z}}$
   (ii) $z^2 e^{1/z}$
   (iii) $\frac{1}{z-\sin z}$.

   Justify your answers.

   (c) Show that if $f$ has an isolated singularity at a point $z_0$ and $f$ is bounded on a neighbourhood of $z_0$, then $z_0$ is a removable singularity.

   You may use without proof that if $f$ is holomorphic on a disc whose boundary is $\gamma$ then $\int_\gamma \frac{f(w)}{w-z}\,dw$ is also holomorphic.

   (d) State Casorati - Weierstrass' theorem about the range of a holomorphic function in a neighbourhood of an essential singularity.

6. (a) Where are the poles of

$$\frac{1}{(z^2+1)^2(z^2+4)}$$

   and what is the order of each?

   (b) By integrating along a large half-circle, evaluate

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2(x^2+4)}\,dx.$$